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Stability in a Two-Dimensional Dynamical System of Endogenous Growth with Public Capital

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Abstract:
The aim of this study is to present a stability in a two-dimensional dynamical system of endogenous growth with public capital. We assume the simple model of the economic growth, in which both private and public capital can influence on the rate of growth of knowledge. The public capital is rival but non excludable goods, i.e. there is a congestion in use of public capital. The model of growth is formulated as a two-dimensional dynamical system. Using mathematical methods of dynamical systems, we analyze growth paths as well as the stationary states of the system and their stability.
Keywords: economic growth, endogenous growth, public capital.

1. Introduction

The Solow model [37] describes the dynamics of a simple economy with one input – physical capital. There is a natural way to extend this model including the other forms of capitals, especially human capital [30]. Another choice is the public capital.

There is an increasing role of the public sector in economies and the study of government policy is a central topic of economic dynamical analyses. Public sector economics focuses on budgetary receipts (taxes) and budget expenditure [6]. The purpose of public spending is not only consumption, production of goods, but also investments. Physical public capital is treated as a resource of materials used in the production process. Physical public capital affects the production capacity of the entire economy. At work, we will limit to the analysis of public investment, which results in capital goods, a factor of production such as private capital in kind. In this paper we concentrate on the dynamical analysis of a simple model of economic growth with public capital.

Public capital is set of assets that are own the government and are used for productivity. Examples of public capital are highways, water systems, sewers, airports, roads, transit systems railways, public education, public hospitals, police and fire protection, prisons, and courts, public electric and gas utilities, and telecommunications [3].

Public sector capital stock has an impact on private sector production. But on the other hand provision of public sector capital has little effect on private firms' production possibilities [34]. Government can benefit from a more efficient allocation of public resources to attain a higher growth rate [35].

In the earliest models of economic growth public investment was treated as the public expenditure. Barro discussed the model in which the government expenditures raise the marginal productivity of private capital [8]. Then Barro and Sala-i-Martin considered models with rival and excludable publicly provided private goods, non-rival and non-excludable publicly-provided public goods and publicly-provided goods which are the subject of congestion [9].

The alternative approach presented by Greiner who assumed that public inputs are capital goods accumulated in a similar manner as private physical capital [18]. The golden rule of public finances says to finance only long-term investment expenditures with public debt. Greiner [20] proved that greater indebtedness goes hand in hand with smaller long-term growth, obtaining the condition that the increased public spending deficit increases the growth rate. In addition to models of economic growth with a balanced budget, models with a budget deficit will be examined.

We follow Greiner's approach and make two additional assumptions. First, we assume the congestion of the public capital. It means that, for a given level of public capital, the increase of private capital decreases the quantity of public capital to every firm. The production function is of the form proposed by Bajo-Rubio [6]

$$Y = K^{\alpha} Z_1^{\beta_1} \dots Z_m^{\beta_m} (AL)^{1-\alpha-\beta_1-\dots-\beta_m} \left(\frac{G}{K}\right)^{\gamma} \left(\frac{T}{K}\right)^{\theta}, \tag{1}$$

where $\alpha > \gamma + \theta$, and Y denotes output, K is the private physical capital, G is the public physical capital, Z_i are other inputs such as, e.g., human capital or public capital, L is labor, A is knowledge and T is transfer.

The second assumption is the dependence of the rate of change of knowledge on the rates of change of capital inputs [36]

$$\frac{\dot{A}}{A} = a + \mu\frac{\dot{K}}{K} + \nu_1\frac{\dot{Z}_1}{Z_1} + \dots + \nu_m\frac{\dot{Z}_m}{Z_m} + \nu_{m+1}\frac{\dot{G}}{G}. \tag{2}$$

To analyze the dynamics of the model we use the methods of dynamical system theory [31]. These method are especially suitable to study the dynamics of nonlinear economic systems [27]. They allow to determine qualitatively the regions of initial conditions for which trajectories reach the same asymptotic state (steady-state point). Qualitative behavior of dynamical systems depend on the values of model parameters. The change of behavior due to the change of the value of parameter is a scope of the bifurcation theory [13, 22].

Apart from analytical methods of dynamical systems, we use numerical methods of integration of differential equations. This allows us to present phase portraits of system under study as well as investigate the bifurcations in details.

2. The Model

We consider the economy where output Y is produced by using physical private capital K, physical public capital G, labor L, and knowledge A as inputs

$$Y(t) = F(K(t),G(t),A(t),L(t)). \tag{3}$$

We assume the neoclassical production function, proposed by Bajo-Rubio [6], in the simplified form

$$Y(t) = K^{\alpha}(t)[A(t)L(t)]^{1-\alpha}\left[\frac{G(t)}{K(t)}\right]^{\beta}, \tag{4}$$

where $0 < \beta < \alpha < 1$.

We make the standard assumption that the labor grows with the constant rate n

$$\frac{\dot{L}(t)}{L(t)} = n. \tag{5}$$

In the neoclassical model of economic growth it is assumed that knowledge A grows with the constant rate a. We relax this assumption and assume that apart from the exogenous growth of knowledge both private and public capital can influence on the rate of growth of knowledge. We assume that these processes are additive and proportional to rates of growth of these capitals

$$\frac{\dot{A}(t)}{A(t)} = a + \mu\frac{\dot{K}(t)}{K(t)} + \nu\frac{\dot{G}(t)}{G(t)} \tag{6}$$

or

$$A(t) = A_0 e^{at} K^{\mu}(t) G^{\nu}(t), \tag{7}$$

where μ is rate of growth of private capital and ν is rate of growth of public capital. The change of the private capital is equal to the net investment

$$\dot{K}(t) = s(1-\tau)Y(t) - \delta K(t), \tag{8}$$

while the change of the public capital is given by

$$\dot{G}(t) = \tau Y(t) - \delta G(t), \tag{9}$$

where we assume that both private and public capital depreciate with the rate δ, and s and τ are the rates of saving and tax, respectively. Let us introduce the new variables

$$k = \frac{K}{AL}, \quad g = \frac{G}{AL}. \tag{10}$$

Then the system of equations (5), (6), (8), (9) can be reduced to the two - dimensional dynamical system

$$\dot{k}(t) = s(1-\tau)(1-\mu)k^{\alpha-\beta}(t)g^{\beta}(t) - \tau\nu k^{\alpha-\beta+1}(t)g^{\beta-1}(t) - dk(t) \tag{11a}$$
$$\dot{g}(t) = \tau(1-\nu)k^{\alpha-\beta}(t)g^{\beta}(t) - s(1-\tau)\mu k^{\alpha-\beta-1}(t)g^{\beta+1}(t) - dg(t) \tag{11b}$$

where

$$d = (1-\mu-\nu)\delta + n + a. \tag{12}$$

3. Local Stability Analysis

For the two-dimensional dynamical system in the form

$$\frac{dx}{dt} \equiv \dot{x} = P(x,y)$$
$$\frac{dy}{dt} \equiv \dot{y} = Q(x,y)$$

The critical point (x^*,y^*) is determined as a solution of the system

$$P(x,y) = 0, \tag{14a}$$
$$Q(x,y) = 0. \tag{14b}$$

To determine the character of the critical point (x^*,y^*), first, we find the linearization matrix

$$A = \begin{bmatrix} \frac{dP(x,y)}{dx}\Big|_{x=x^*,y=y^*} & \frac{dP(x,y)}{dy}\Big|_{x=x^*,y=y^*} \\ \frac{dQ(x,y)}{dx}\Big|_{x=x^*,y=y^*} & \frac{dQ(x,y)}{dy}\Big|_{x=x^*,y=y^*} \end{bmatrix} \tag{15}$$

and then we solve the characteristic equation

$$\lambda^2 - (\mathrm{tr}A)\lambda + \det A = 0, \tag{16}$$

where λ is an eigenvalue of the linearization matrix A, and tr A and det A are the trace and the determinant of matrix A, respectively.

The eigenvalues can be real or complex (positive or negative discriminant $(\mathrm{tr}\,A)^2 - 4\det A$). If the eigenvalues are real, different signs, the critical point is a saddle. If the eigenvalues are real of the same sign the critical point is a node (stable for negative eigenvalues or unstable for positive eigenvalues). If the eigenvalues are complex with a zero real part, the critical point is a center. If the eigenvalues are complex with a non-zero real part, the critical point is a focus (stable for negative real part of the eigenvalue or unstable for positive real part of the eigenvalue).

There are two critical points of the system (11) in the finite domain of the phase space. The first, trivial point is

$$k_1^* = 0, \qquad g_1^* = 0. \tag{17}$$

The second critical point lies inside the domain $k > 0$ and $g > 0$ and it is the only critical point in this domain and it is an attractor for all trajectories with initial conditions in this domain

$$k^* = \left(\frac{d}{\tau(1-\mu-\nu)}\right)^{\frac{1}{\alpha-1}} \left(\frac{s(1-\tau)}{\tau}\right)^{\frac{\beta-1}{\alpha-1}}, \tag{18a}$$

$$g^* = \left(\frac{d}{\tau(1-\mu-\nu)}\right)^{\frac{1}{\alpha-1}} \left(\frac{s(1-\tau)}{\tau}\right)^{\frac{\beta-\alpha}{\alpha-1}}. \tag{18b}$$

To determine the character of critical point 18, we calculate the linearization matrix at this point

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{19}$$

where

$$a_{11} = d\left(\frac{(1-\mu)(\alpha-\beta) - \nu(\alpha-\beta+1)}{1-\mu-\nu} - 1\right), \tag{20a}$$

$$a_{12} = \frac{ds(1-\tau)}{(1-\nu-\mu)\tau}\left[(1-\mu)\beta - (\beta-1)\nu\right], \tag{20b}$$

$$a_{21} = \frac{\tau d}{s(1-\mu-\nu)(1-\tau)}\left[(1-\nu)(\alpha-\beta) - \mu(\alpha-\beta-1)\right], \tag{20c}$$

$$a_{22} = d\left(\frac{\beta(1-\nu) - \mu(\beta+1)}{1-\mu-\nu} - 1\right). \tag{20d}$$

Solving the characteristic equation we find that there are two distinct real negative eigenvalues. This point is a stable node.

$$\lambda_1 = -\frac{d}{1-\mu-\nu} \tag{21}$$

$$\lambda_2 = d(\alpha-1). \tag{22}$$

4. Saddle-node Bifurcation

In this section we study local bifurcation in the system (23). Assume that we have a two-dimensional dynamical system

$$\dot{k}(t) = s(1-\tau)(1-\mu)k^{\alpha-\beta}(t)g^{\beta}(t) - \tau\nu k^{\alpha-\beta+1}(t)g^{\beta-1}(t) - dk(t), \tag{23a}$$

$$\dot{g}(t) = \tau(1-\nu)k^{\alpha-\beta}(t)g^{\beta}(t) - s(1-\tau)\mu k^{\alpha-\beta-1}(t)g^{\beta+1}(t) - dg(t). \tag{23b}$$

We solve the characteristic equation

$$\lambda^2 - (\mathrm{tr}\, A)\lambda + \det A = 0. \tag{24}$$

In our case:

$$\det A = \frac{d^2(1-\alpha)}{1-\mu-\nu}, \tag{25}$$

$$\mathrm{tr}\, A = \frac{d}{1-\nu-\mu}((\alpha-2)(1-\nu-\mu) - (\mu+\nu)). \tag{26}$$

Proposition 1 *The saddle-node bifurcation arises if and only if* $\det A = 0$.

In this case $\frac{d^2(1-\alpha)}{1-\mu-\nu} = 0$ only if $\nu = 1 - \mu + \frac{n+a}{\delta}$ and $1 \neq \mu + \nu$.

Proposition 2 *In the case of a two-dimentional system, a Hopf bifurcation generally arises if and only if* det $A = tr\ A$. *The saddle-node bifurcation arises if and only if* det $A = tr\ A$.

Hopf bifurcation does not appear in this model.

5. Conclusions

In this paper we considered the model of economic growth with public capital to present an impact of the public capital and knowledge on the economic growth from a theoretical perspective. we consider the economy where output is produced by using private capital, public capital, labor and knowledge as inputs. In the model we assume that both public and private capital can influence on the rate of growth of knowledge. In the work we consider that for a given level of public capital, the increase of private capital decreases the quantity of public capital to every firm.

1) The dynamics of the model can be represented as a two-dimensional dynamical system in variables: a ratio of rate of public and private capital torate of knowledge, tax and level of labor. We obtained the critical point of the model is a stable node.
2) In the model we find the saddle-node bifurcation. Due to this bifurcation, the saddle critical point is created toward which the system evolves along the stable optimal path.

6. The Further Research

In this project we are going to extend the study the model, in particular

- we study the dependence of model solution on the model parameters; we use the methods of bifurcation analysis to this aim;
- we consider the Ramsey problem in this model; as a result we obtain the three-dimensional dynamical system which will be a subject to a thorough scrutiny;
- the numerical analysis of the models will be made.

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